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Research Article

On the Fixed-Point Property of Unital Uniformly Closed Subalgebras of $C(X)$

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Let X be a compact Hausdorff topological space and let $C(X)$ and $C_{\mathbb{R}}(X)$ denote the complex and real Banach algebras of all continuous complex-valued and continuous real-valued functions on X under the uniform norm on X , respectively. Recently, Fupinwong and Dhompongsa (2010) obtained a general condition for infinite dimensional unital commutative real and complex Banach algebras to fail the fixed-point property and showed that $C_{\mathbb{R}}(X)$ and $C(X)$ are examples of such algebras. At the same time Dhompongsa et al. (2011) showed that a complex C^* -algebra A has the fixed-point property if and only if A is finite dimensional. In this paper we show that some complex and real unital uniformly closed subalgebras of $C(X)$ do not have the fixed-point property by using the results given by them and by applying the concept of peak points for those subalgebras.

1. Introduction and Preliminaries

We let \mathbb{C} , \mathbb{R} , $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ denote the fields of complex, real numbers, the set of natural numbers, the unit circle, the open unit disc, and the closed unit disc, respectively. The symbol \mathbb{F} denotes a field that can be either \mathbb{C} or \mathbb{R} . The elements of \mathbb{F} are called scalars.

Let X be a compact topological space. We denote by $C_{\mathbb{F}}(X)$ the unital commutative Banach algebra (over \mathbb{F}) of continuous functions from X to \mathbb{F} with pointwise addition, scalar multiplication, and product with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \quad (f \in C_{\mathbb{F}}(X)). \quad (1.1)$$

For applying the usual notation, we write $C(X)$ instead of $C_{\mathbb{C}}(X)$.

Let $T : E \rightarrow E$ be a self-map on the nonempty set E . We denote $\{x \in E : T(x) = x\}$ by $\text{Fix}(T)$ and call the *fixed-points set* of T .

Let \mathfrak{X} be a normed space over the field \mathbb{F} . A mapping $T : E \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ is *nonexpansive* if $\|T(f) - T(g)\| \leq \|f - g\|$ for all $f, g \in E$. We say that the normed space \mathfrak{X} has the *fixed-point property* if for every nonempty bounded closed convex subset E of \mathfrak{X} and every *nonexpansive* mapping $T : E \rightarrow E$ we have $\text{Fix}(T) \neq \emptyset$. One of the central goals in fixed point theory is to find which Banach spaces have the fixed-point property.

Let A be a unital algebra (over \mathbb{F}) with unit 1 and let $G(A)$ denote the set of all invertible elements of A . We define the *spectrum* of an element f of A to be the set $\{\lambda \in \mathbb{F} : \lambda 1 - f \notin G(A)\}$ and denote it by $\sigma(f)$. The *spectral radius* of f , denoted by $r(f)$, is defined to be $\sup\{|\lambda| : \lambda \in \sigma(f)\}$. Note that if A is a unital complex Banach algebra, then $r(f) = \lim_{n \rightarrow \infty} \|f^n\|^{1/n} = \inf\{\|f^n\|^{1/n} : n \in \mathbb{N}\}$ (see [1, Theorem 10.13]).

A *character* on a unital algebra A over \mathbb{F} is a nonzero homomorphism $\varphi : A \rightarrow \mathbb{F}$. We denote by $\Omega(A)$ the set of all characters on A . If A is a unital commutative complex Banach algebra, $\Omega(A) \neq \emptyset$ and $\sigma(f) = \{\varphi(f) : \varphi \in \Omega(A)\}$ for all $f \in A$ (see [2, 3]). Note that if A is real algebra, it may be the case that $\Omega(A) = \emptyset$ (see [4, Example 2.4] and Example 3.9 below) or $\Omega(A) \neq \emptyset$ and $\sigma(f) \neq \{\varphi(f) : \varphi \in \Omega(A)\}$ (see Example 3.8 below).

Let A be a unital commutative real Banach algebra. A *complex character* on A is a nonzero homomorphism $\varphi : A \rightarrow \mathbb{C}$, regarded as a real algebra. The set of all complex character on A is called the *carrier space* of A and denoted by $\text{Car}(A)$. Obviously, $\Omega(A) \subseteq \text{Car}(A)$.

Let X be a compact topological space and let A be a unital uniformly closed subalgebra of $C_{\mathbb{F}}(X)$. For each $x \in X$, the map $\varepsilon_x : A \rightarrow \mathbb{F}$ defined by $\varepsilon_x(f) = f(x)$, belongs to $\Omega(A)$ which is called the *evaluation character* on A at x . It is known that $\Omega(C(X)) = \{\varepsilon_x : x \in X\}$.

Let \mathcal{F} be a collection of complex-valued functions on a nonempty set X . We say that:

- (i) \mathcal{F} separates the points of X if for each $x, y \in X$ with $x \neq y$, there is a function f in \mathcal{F} such that $f(x) \neq f(y)$;
- (ii) \mathcal{F} is *self-adjoint* if $f \in \mathcal{F}$ implies that $\bar{f} \in \mathcal{F}$;
- (iii) \mathcal{F} is *inverse-closed* if $1/f \in \mathcal{F}$ whenever $f \in \mathcal{F}$ and $f(x) \neq 0$ for all $x \in X$.

Let A be a unital commutative complex Banach algebra. It is known that each $\varphi \in \Omega(A)$ is continuous and $\|\varphi\| = 1$. For each $f \in A$, we define the map $\hat{f} : \Omega(A) \rightarrow \mathbb{C}$ by $\hat{f}(\varphi) = \varphi(f)$ ($\varphi \in \Omega(A)$) and say that \hat{f} is the *Gelfand transform* of f . We denote the set $\{\hat{f} : f \in A\}$ by \hat{A} . It is easy to see that \hat{A} separates the points of $\Omega(A)$. The *Gelfand topology* of $\Omega(A)$ is the weakest topology on $\Omega(A)$ for which every $\hat{f} \in \hat{A}$ is continuous. In fact, the Gelfand topology of $\Omega(A)$ coincides with the relative topology on $\Omega(A)$ which is given by weak* topology of A^* , the dual space of A . We know that $\Omega(A)$ with the Gelfand topology is a compact Hausdorff topological space and \hat{A} is a complex subalgebra of $C(\Omega(A))$ (see [1, 3]). Clearly, the following statements are equivalent.

- (i) \hat{A} is self-adjoint.
- (ii) For each $f \in A$, there exists an element $g \in A$ such that $\varphi(g) = \overline{\varphi(f)}$ for all $\varphi \in \Omega(A)$.

Let X be a topological space. A self-map $\tau : X \rightarrow X$ is called a *topological involution* on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Let X be a compact Hausdorff topological space and τ be a topological involution on X . We denote by $C(X, \tau)$ the set of all $f \in C(X)$ for which $\bar{f} \circ \tau = f$. Then $C(X, \tau)$ is a unital uniformly closed real subalgebra of $C(X)$ which separates the points of X , does not contain the constant function i and we have $C(X) = C(X, \tau) \oplus iC(X, \tau)$. Moreover, $C(X, \tau) = C_{\mathbb{R}}(X)$ if and only if τ is the identity

map on X . Let A be a unital uniformly closed real subalgebra of $C(X, \tau)$. For each $x \in X$ the map $e_x : A \rightarrow \mathbb{C}$ defined by $e_x(f) = f(x)$, is a complex character on A which is called the *evaluation complex character* on A at x . We know that $\text{Car}(C(X, \tau)) = \{e_x : x \in X\}$ (see [5]). The algebra $C(X, \tau)$ was first introduced by Kulkarni and Limaye in [6]. We denote by $C_{\mathbb{R}}(X, \tau)$ the set of all $f \in C(X, \tau)$ for which f is real-valued on X . Then $C_{\mathbb{R}}(X, \tau)$ is a unital uniformly closed real subalgebra of $C(X, \tau)$.

Let X be a compact Hausdorff topological space and let A be a unital real or complex subspace of $C(X)$. A nonempty subset P of X called a *peak set* for A if there exists a function f in A such that $P = \{x \in X : f(x) = 1\}$ and $|f(y)| < 1$ for all $y \in X \setminus P$, the function f is said to peak on P . If the peak set P for A is the singleton $\{x\}$, we call x a *peak point* for A . The set of all peak points for A is denoted by $S_0(A, X)$. A nonempty subset E of X is called a *boundary* for A , if for each $f \in A$ there is an element x of E such that $\|f\|_X = |f(x)|$. Clearly, $S_0(A, X) \subseteq E$ whenever E is a boundary for A . It is known that, if X is a first countable compact Hausdorff topological space then $S_0(C(X), X) = X$ (see [7]).

Let τ be a topological involution on a compact Hausdorff topological space X and let A be a unital uniformly closed real subspace of $C(X, \tau)$. If $P \subseteq X$ is a peak set for A , then $\tau(P) = P$.

Definition 1.1. Let τ be a topological involution on a compact Hausdorff topological space X and A be a unital uniformly closed real subspace of $C(X, \tau)$. We say that $x \in X$ is a τ -*peak point* for A if $\{x, \tau(x)\}$ is a peak set for A . We denote by $T_0(A, X, \tau)$ the set of all τ -peak points for A .

Let X be a compact Hausdorff topological space and τ be a topological involution on X . Let B be a unital uniformly closed subalgebra of $C(X)$ such that $\bar{f} \circ \tau \in B$ for all $f \in B$ and define $A = \{f \in B : \bar{f} \circ \tau = f\}$. Then A is a unital uniformly closed real subalgebra of $(C(X, \tau))$, $B = A \oplus iA$, $S_0(A, X) = S_0(B, X) \cap \text{Fix}(\tau)$ and $T_0(A, X, \tau) = S_0(B, X)$ (see [5]).

Fupinwong and Dhompongsa studied the fixed-point property of unital commutative Banach algebras over field \mathbb{F} in [4]. In the case $\mathbb{F} = \mathbb{R}$, they obtained the following results.

Theorem 1.2 (see [4, Theorem 3.1]). *Let A be an infinite dimensional unital commutative real Banach algebra satisfying each of the following:*

- (i) $\Omega(A) \neq \emptyset$ and $\sigma(f) = \{\varphi(f) : f \in \Omega(A)\}$,
- (ii) if $f, g \in A$ such that $|\varphi(f)| \leq |\varphi(g)|$ for each $\varphi \in \Omega(A)$, then $\|f\| \leq \|g\|$,
- (iii) $\inf\{r(f) : f \in A, \|f\| = 1\} > 0$.

Then A does not have the fixed-point property.

Theorem 1.3 (see [4, Corollary 3.2]). *Let X be a compact Hausdorff topological space. If $C_{\mathbb{R}}(X)$ is infinite dimensional, then $C_{\mathbb{R}}(X)$ fails to have the fixed-point property.*

In the case $\mathbb{F} = \mathbb{C}$, they obtained the following result.

Theorem 1.4 (see [4, Theorem 4.3]). *Let A be an infinite dimensional unital commutative complex Banach algebra satisfying each of the following:*

- (i) \hat{A} is self-adjoint,
- (ii) if $f, g \in A$ such that $|\varphi(f)| \leq |\varphi(g)|$ for each $\varphi \in \Omega(A)$, then $\|f\| \leq \|g\|$,
- (iii) $\inf\{r(f) : f \in A, \|f\| = 1\} > 0$.

Then A does not have the fixed-point property.

By using the above theorem, we obtain the following result.

Theorem 1.5. *Let X be a compact Hausdorff topological space. If $C(X)$ is infinite dimensional, then $C(X)$ fails to have the fixed-point property.*

Dhompongsa et al. studied the fixed-point property of complex C^* -algebras in [8] and obtained the following result.

Theorem 1.6 (see [8, Theorem 1.4]). *The following properties for a complex C^* -algebras A are equivalent:*

- (i) A has the fixed-point property;
- (ii) A has finite dimension.

In this paper, we give a general condition for some infinite dimensional unital uniformly closed subalgebras of $C(X)$ to fail the fixed-point property by applying Theorems 1.4 and 1.6. By using the concept of peak points for unital uniformly closed subalgebras of $C(X)$, we show that some of these algebras do not have the fixed-point property. We also prove that $C_{\mathbb{R}}(X, \tau)$ and $C(X, \tau)$ fail to have the fixed-point property. By using the concept of τ -peak points for unital uniformly closed real subalgebras of $C(X, \tau)$, we show that some of these algebras do not have the fixed-point property.

2. FPP of Complex Subalgebras of $C(X)$

We first obtain a general condition for infinite dimensional unital uniformly closed subalgebra of $C(X)$ to fail the fixed-point property and give an infinite collection of these algebras.

Theorem 2.1. *Let X be a compact topological space. If A is a infinite dimensional self-adjoint uniformly closed subalgebras of $C(X)$, then A does not have the fixed-point property.*

Proof. By hypothesis, A is an infinite dimensional complex C^* -algebra under the natural involution $f \mapsto \bar{f} : A \rightarrow A$. Then, A does not have the fixed-point property by Theorem 1.6. \square

Example 2.2. Let $m \in \mathbb{N}$ and let A_m be the uniformly closed subalgebra of $C(\mathbb{T})$ generated by $1, Z^{2m}$ and \overline{Z}^{2m} , where Z is the coordinate function on \mathbb{T} . Then A_m is an infinite dimensional self-adjoint uniformly closed subalgebra of $C(\mathbb{T})$ and so A_m does not have the fixed-point property.

Proof. It is easy to see that A_m is self-adjoint. To complete the proof, it is enough to show that A_m is infinite dimensional. We define the sequence $\{f_{m,n}\}_{n=0}^{\infty}$ of elements of A_m by

$$f_{m,0} = 1, \quad f_{m,n} = Z^{2^n m} - 1 \quad (n \in \mathbb{N}). \quad (2.1)$$

We can prove that for each $n \in \mathbb{N}$ the set $\{f_{m,0}, f_{m,1}, \dots, f_{m,n}\}$ is a linearly independent set of elements of A_m . Therefore, A_m is infinite dimensional. \square

We now show that some of the unital uniformly closed subalgebras of $C(X)$ fail to have the fixed-point property by using the concept of peak points for these algebras.

Theorem 2.3. *Let X be a compact Hausdorff topological space and let A be a unital uniformly closed complex subalgebra of $C(X)$. If $S_0(A, X)$ contains a limit point of X , then A does not have the fixed-point property.*

Proof. Let $x_0 \in S_0(A, X)$ be a limit point of X . Then there exists a function $f_0 \in A$ with $f_0(x_0) = 0$ and $|f_0(x)| < 1$ for all $x \in X \setminus \{x_0\}$, and there exists a net $\{x_\alpha\}_\alpha$ in $X \setminus \{x_0\}$ such that $\lim_\alpha x_\alpha = x_0$ in X . We define $E = \{f \in A : \|f\|_X = f(x_0) = 1\}$. Then E is a nonempty bounded closed convex subset of A and $f_0 f \in E$ for all $f \in E$. We define the map $T : E \rightarrow E$ by $T(f) = f_0 f$. It is easy to see that T is a nonexpansive mapping on E .

We claim that $\text{Fix}(T) = \emptyset$. Suppose $f_1 \in \text{Fix}(T)$. Then $f_0 f_1 = f_1$ and so $f_1(x) = 0$ for all $x \in X \setminus \{x_0\}$. The continuity of f_1 in x_0 implies that $\lim_\alpha f_1(x_\alpha) = f_1(x_0)$. Therefore, $f_1(x_0) = 0$, contradicting to $f_1 \in E$. Hence, our claim is justified. Consequently, A does not have the fixed-point property. \square

Corollary 2.4. *Let X be a perfect compact Hausdorff topological space. If A is a unital uniformly closed subalgebra of $C(X)$ with $S_0(A, X) \neq \emptyset$, then A does not have the fixed-point property.*

Example 2.5. Let $A(\overline{\mathbb{D}})$ denote the disk algebra, the complex Banach algebra of all continuous complex-valued functions on $\overline{\mathbb{D}}$ which are analytic on \mathbb{D} under the uniform norm $\|f\|_{\overline{\mathbb{D}}} = \sup\{|f(z)| : z \in \overline{\mathbb{D}}\}$ ($f \in A(\overline{\mathbb{D}})$). Then $A(\overline{\mathbb{D}})$ does not have the fixed-point property.

Proof. Clearly $\overline{\mathbb{D}}$ is a perfect compact Hausdorff topological space and $A(\overline{\mathbb{D}})$ is a unital uniformly closed complex subalgebra of $C(\overline{\mathbb{D}})$. By the principle of maximum modulus, $S_0(A(\overline{\mathbb{D}}), \overline{\mathbb{D}}) \subseteq \mathbb{T}$. Now let $\lambda \in \mathbb{T}$. It is easy to see that the function $f : \overline{\mathbb{D}} \rightarrow \mathbb{C}$, defined by $f(z) = (1/2)(1 + \bar{\lambda}z)$, belongs to $A(\overline{\mathbb{D}})$ and peaks at λ . Therefore, $S_0(A(\overline{\mathbb{D}}), \overline{\mathbb{D}}) = \mathbb{T}$. It follows that $A(\overline{\mathbb{D}})$ does not have the fixed-point property by Corollary 2.4. \square

Now by giving an example we show that the converse of Theorem 2.3 is not necessarily true, in general.

Example 2.6. Let J be an uncountable set and let X_α be the unit closed interval $[0, 1]$ with the standard topology for each $\alpha \in J$. Suppose $X = \prod_{\alpha \in J} X_\alpha$ with the product topology. Then $C(X)$ fails to have the fixed-point property but $S_0(C(X), X) = \emptyset$ and so $S_0(C(X), X)$ does not contain any limit points of X .

Proof. Clearly, X is an infinite compact Hausdorff topological space. Choose a sequence $\{x_n\}_{n=1}^\infty$ in X such that $x_j \neq x_k$, where $j, k \in \mathbb{N}$ and $j \neq k$. By Urysohn's lemma, there exists a sequence $\{h_n\}_{n=1}^\infty$ in $C(X)$ such that $h_1 = 1$ and $h_n(x_1) = \cdots = h_n(x_{n-1}) = 0$, $h_n(x_n) = 1$ for all $n \geq 2$. It is easy to see that the set $\{h_1, \dots, h_n\}$ is a linearly independent set in $C(X)$ for all $n \in \mathbb{N}$. Thus, $C(X)$ is an infinite dimensional complex vector space. Therefore, $C(X)$ does not have the fixed-point property by Theorem 1.5.

We now show that $S_0(C(X), X) = \emptyset$. We assume that E is the set of all $\underline{x} = (x_\alpha)_{\alpha \in J} \in X$ for which there is a countable subset $I_{\underline{x}}$ of J such that $x_\alpha = 0$ for all $\alpha \in J \setminus I_{\underline{x}}$ and F is the set of all $\underline{x} = (x_\alpha)_{\alpha \in J} \in X$ for which there is a countable subset $J_{\underline{x}}$ of J such that $x_\alpha = 1$ for all $\alpha \in J \setminus J_{\underline{x}}$. Clearly, $E \cap F = \emptyset$. It is easy to see that E and F are boundaries for $C(X)$. Therefore, $S_0(C(X), X) = \emptyset$. \square

Remark 2.7. Let X be an infinite first countable compact Hausdorff topological space. Then $S_0(C(X), X) = X$, and X has at least one limit point. Hence $S_0(C(X), X)$ contains a limit point of X . Therefore, $C(X)$ fails to have the fixed-point property by Theorem 2.3.

3. FPP of Real Subalgebras of $C(X)$

We first give a sufficient condition for unital uniformly closed real subalgebras of $C_{\mathbb{R}}(X)$ to fail the fixed-point property.

Lemma 3.1. *If A is a unital commutative real Banach algebra with $\Omega(A) \neq \emptyset$, then $\{\varphi(f) : \varphi \in \Omega(A)\} \subseteq \sigma(f)$ for all $f \in A$.*

Proof. Let $f \in A$. For each $\varphi \in \Omega(A)$, we define $g_{\varphi} = \varphi(f)1 - f$. Then $g_{\varphi} \in A$ and $\varphi(g_{\varphi}) = 0$. Therefore, $g_{\varphi} \notin G(A)$ and so $\varphi(f) \in \sigma(f)$. \square

Lemma 3.2. *Let X be a compact topological space. If A is an inverse closed unital uniformly closed real subalgebra of $C_{\mathbb{R}}(X)$, then $\Omega(A) \neq \emptyset$, $\Omega(A) = \{\varepsilon_x : x \in X\}$ and $\sigma(f) = \{\varphi(f) : \varphi \in \Omega(A)\}$ for all $f \in A$.*

Proof. Since A is a unital real subalgebra of $C_{\mathbb{R}}(X)$, $\varepsilon_x \in \Omega(A)$ for all $x \in X$. Therefore, $\Omega(A) \neq \emptyset$ and so $\{\varphi(f) : \varphi \in \Omega(A)\} \subseteq \sigma(f)$ for all $f \in A$ by Lemma 3.1.

Now, let $f \in A$ and let $\lambda \in \mathbb{C} \setminus \{\varphi(f) : \varphi \in \Omega(A)\}$. Then $\lambda - \varphi(f) \neq 0$ for each $\varphi \in \Omega(A)$, and so $(\lambda 1 - f)(x) \neq 0$ for all $x \in X$. Therefore, $\lambda 1 - f \in G(A)$ because A is inverse-closed. It follows that $\lambda \in \mathbb{C} \setminus \sigma(f)$ and so $\sigma(f) \subseteq \{\varphi(f) : \varphi \in \Omega(A)\}$. We now show that $\Omega(A) \subseteq \{\varepsilon_x : x \in X\}$. Suppose $\varphi \in \Omega(A) \setminus \{\varepsilon_x : x \in X\}$. Let $x \in X$. Then there exists a function f_x in A such that $\varphi(f_x) \neq f_x(x)$. We define $g_x = f_x - \varphi(f_x)1$. Then $g_x \in A$, $\varphi(g_x) = 0$ and $g_x(x) \neq 0$. The continuity of g_x on X implies that there exists a neighborhood U_x of x in X such that $g_x(y) \neq 0$ for all $y \in U_x$. By compactness of X , there exist finite elements x_1, \dots, x_m of X such that $X = \bigcup_{j=1}^m U_{x_j}$. Define $g = \sum_{j=1}^m (g_{x_j})^2$. Clearly, $g \in A$ and $\varphi(g) = 0$. Moreover, $g(y) \neq 0$ for all $y \in X$. Since A is inverse-closed, $1/g \in A$. It follows that $\varphi(g) \neq 0$. This contradiction implies that $\Omega(A) \subseteq \{\varepsilon_x : x \in X\}$. \square

Theorem 3.3. *Let X be a compact topological space. If A is an infinite dimensional inverse-closed unital uniformly closed real subalgebra of $C_{\mathbb{R}}(X)$, then A does not have the fixed-point property.*

Proof. Since A is a unital uniformly closed real subalgebra of $C_{\mathbb{R}}(X)$, we have $\Omega(A) \neq \emptyset$, $\Omega(A) = \{\varepsilon_x : x \in X\}$ and $\sigma(f) = \{\varphi(f) : \varphi \in \Omega(A)\} = \{f(x) : x \in X\}$ for all $f \in A$ by Lemma 3.2. Therefore, $r(f) = \sup\{|f(x)| : x \in X\} = \|f\|_X$ for all $f \in A$. It follows that $\inf\{r(f) : f \in A, \|f\|_X = 1\} > 0$. Now, let $f, g \in A$ with $|\varphi(f)| \leq |\varphi(g)|$ for all $\varphi \in \Omega(A)$. Then, $|f(x)| \leq |g(x)|$ for each $x \in X$ and so $\|f\|_X \leq \|g\|_X$. Since A is infinite dimensional, we conclude that A does not have the fixed-point property by Theorem 1.2. \square

Proposition 3.4. *Let X be an infinite compact Hausdorff topological space and let τ be a topological involution on X . Then*

(i) $C_{\mathbb{R}}(X, \tau)$ is infinite dimensional;

(ii) $C(X, \tau)$ is infinite dimensional.

Proof. Choose a sequence $\{x_n\}_{n=1}^\infty$ in X such that $x_j \neq x_k$, where $j, k \in \mathbb{N}$ and $j \neq k$. By Urysohn's lemma, there exists a sequence $\{h_n\}_{n=1}^\infty$ in $C_\mathbb{R}(X)$ such that $h_1 = 1$ and $h_n(x_1) = h_n(\tau(x_1)) = \dots = h_n(x_{n-1}) = h_n(\tau(x_{n-1})) = 0$, $h_n(x_n) = h_n(\tau(x_n)) = 1$ for all $n \geq 2$. We define the sequence $\{f_n\}_{n=1}^\infty$ in $C_\mathbb{R}(X, \tau)$ as the following:

$$f_1 = 1, \quad f_n = (h_n \circ \tau)h_n \quad (n \in \mathbb{N}, n \geq 2). \quad (3.1)$$

It is easy to see that the set $\{f_1, \dots, f_n\}$ is a linearly independent set in $C_\mathbb{R}(X, \tau)$ for all $n \in \mathbb{N}$. Therefore, $C_\mathbb{R}(X, \tau)$ is an infinite dimensional real vector space. (ii) Since $C_\mathbb{R}(X, \tau)$ is a real linear subspace of $C(X, \tau)$, we conclude that $C(X, \tau)$ is infinite dimensional by (i). \square

Theorem 3.5. *Let X be an infinite compact Hausdorff topological space and let τ be a topological involution on X . Then $C_\mathbb{R}(X, \tau)$ does not have the fixed-point property.*

Proof. By part (i) of Proposition 3.4, $C_\mathbb{R}(X, \tau)$ is an infinite dimensional real vector space. On the other hand, $C_\mathbb{R}(X, \tau)$ is an inverse-closed unital uniformly closed real subalgebras of $C_\mathbb{R}(X)$. Therefore, $C_\mathbb{R}(X, \tau)$ does not have the fixed-point property by Theorem 3.3. \square

Corollary 3.6. *Let X be an infinite compact Hausdorff topological space and let τ be a topological involution on X . Then $C(X, \tau)$ does not have the fixed-point property.*

Proof. By Theorem 3.5, $C_\mathbb{R}(X, \tau)$ does not have the fixed-point property. Since $(C(X, \tau), \|\cdot\|_X)$ is a real Banach space and $C_\mathbb{R}(X, \tau)$ is a uniformly closed real subspace of $C(X, \tau)$, we conclude that $C(X, \tau)$ does not have the fixed-point property. \square

We now give a characterization of $\Omega(C(X, \tau))$ as the following.

Theorem 3.7. *Let X be an infinite compact Hausdorff topological space and let τ be a topological involution on X .*

- (i) *If $x \in \text{Fix}(\tau)$, then $\varepsilon_x \in \Omega(C(X, \tau))$, where ε_x is evaluation character on $C(X, \tau)$ at x .*
- (ii) *If $\varphi \in \Omega(C(X, \tau))$, there exists $x \in \text{Fix}(\tau)$ such that $\varphi = \varepsilon_x$.*
- (iii) *$\Omega(C(X, \tau)) = \emptyset$ if and only if $\text{Fix}(\tau) = \emptyset$.*

Proof. (i) is obvious. To prove (ii), let $\varphi \in \Omega(C(X, \tau))$. Then $\varphi \in \text{Car}(C(X, \tau))$ and so there exists $x \in X$ such that $\varphi = e_x$, where e_x is the complex character on $C(X, \tau)$ at x . Since $\varphi(C(X, \tau)) \subseteq \mathbb{R}$, we conclude that $f(x) \in \mathbb{R}$ for all $f \in C(X, \tau)$. Therefore, $f(\tau(x)) = f(x)$ for all $f \in C(X, \tau)$. It follows that $x \in \text{Fix}(\tau)$, because $C(X, \tau)$ separates the points of X . Thus $e_x = \varepsilon_x$ and so $\varphi = \varepsilon_x$.

- (iii) This follows from (i) and (ii). \square

Now by giving two examples, we show that there may be a unital commutative real Banach algebra that fails to have the fixed-point property without satisfying any of the conditions of Theorem 1.2.

Example 3.8. Let X be the closed unit interval $[0, 1]$ with the standard topology and let τ be the topological involution on X defined by $\tau(x) = 1 - x$. Since $\text{Fix}(\tau) = \{1/2\}$, we have $\Omega(C(X, \tau)) = \{\varepsilon_{1/2}\}$ by Theorem 3.7. Define the function $f : X \rightarrow \mathbb{C}$ by $f(x) = |1/2 - x|$. Clearly, $f \in C(X, \tau)$ and $f(X) = [0, 1/2]$. If $\lambda \in (-\infty, 1/2) \cup (1, \infty)$, then $\lambda 1 - f \in G(C(X, \tau))$

and so $\lambda \notin \sigma(f)$. On the other hand, $\lambda 1 - f \notin G(C(X, \tau))$ for all $\lambda \in [1/2, 1]$. Therefore, $\sigma(f) = [1/2, 1]$. But

$$\{\varphi(f) : \varphi \in \Omega(C(X, \tau))\} = \{\varepsilon_{1/2}(f)\} = \left\{f\left(\frac{1}{2}\right)\right\} = \{0\}. \quad (3.2)$$

Thus $\sigma(f) \neq \{\varphi(f) : \varphi \in \Omega(C(X, \tau))\}$. This shows that $C(X, \tau)$ does not satisfy in the condition (i) of Theorem 1.2, but $C(X, \tau)$ fail to have the fixed-point property by Corollary 3.6.

Example 3.9. Let $X = [-2, -1] \cup [1, 2]$ with standard topology and let τ be the topological involution on X defined by $\tau(x) = -x$. Since $\text{Fix}(\tau) = \emptyset$, we have $\Omega(C(X, \tau)) = \emptyset$ by Theorem 3.7. It shows that $C(X, \tau)$ does not satisfy the condition (i) of Theorem 1.2, but $C(X, \tau)$ fails to have the fixed-point property by Corollary 3.6.

We now show that some of the unital closed real subalgebras of $C(X, \tau)$ fails to have the fixed-point property by applying the concept of τ -peak points for these algebras.

Theorem 3.10. *Let X be a compact Hausdorff topological space and let τ be a topological involution on X . Suppose A is a unital uniformly closed real subalgebra of $C(X, \tau)$. If $T_0(A, X, \tau)$ contains a limit point of X , then A does not have the fixed-point property.*

Proof. Let $x_0 \in T_0(A, X, \tau)$ be a limit point of X . Then there exists a function f_0 in A with $f_0(x_0) = f_0(\tau(x_0)) = 1$ and $|f_0(x)| < 1$ for all $x \in X \setminus \{x_0, \tau(x_0)\}$, and there exists a net $\{x_\alpha\}_\alpha$ in $X \setminus \{x_0, \tau(x_0)\}$ such that $\lim_\alpha x_\alpha = x_0$ in X . We define $E = \{f \in A : \|f\|_X = f(x_0) = 1\}$. Then E is a nonempty bounded closed convex subset of A and $f_0 f \in E$ for all $f \in E$. We define the map $T : E \rightarrow E$ by $T(f) = f_0 f$. It is easy to see that T is a nonexpansive mapping on E .

We claim that $\text{Fix}(T) = \emptyset$. Suppose $f_1 \in \text{Fix}(T)$. Then $f_0 f_1 = f_1$ and so $f_1(x) = 0$ for all $x \in X \setminus \{x_0, \tau(x_0)\}$. The continuity of f_1 in x_0 implies that $\lim_\alpha f_1(x_\alpha) = f_1(x_0)$. Therefore, $f_1(x_0) = 0$, contradicting to $f_1 \in E$. Hence, our claim is justified. Consequently, A does not have the fixed-point property. \square

Example 3.11. Let τ be the topological involution on $\overline{\mathbb{D}}$ defined by $\tau(z) = \bar{z}$. We denote by $A(\overline{\mathbb{D}}, \tau)$ the set all $f \in A(\overline{\mathbb{D}})$ for which $\bar{f} \circ \tau = f$. Then $A(\overline{\mathbb{D}}, \tau)$ is a unital uniformly closed real subalgebra of $C(\overline{\mathbb{D}})$ and $A(\overline{\mathbb{D}}) = A(\overline{\mathbb{D}}, \tau) \oplus iA(\overline{\mathbb{D}}, \tau)$. By Example 2.5,

$$T_0(A(\overline{\mathbb{D}}, \tau), \overline{\mathbb{D}}, \tau) = S_0(A(\overline{\mathbb{D}}), \overline{\mathbb{D}}) = \mathbb{T}. \quad (3.3)$$

Therefore, $A(\overline{\mathbb{D}}, \tau)$ does not have the fixed-point property by Theorem 3.10.

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